On the stability Analysis of Uncertain Optimal Control Model for Management of Net Risky Capital Asset

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Abstract

Uncertain process is used in modeling uncertain occurrences that vary with time. The uncertain processes was used to study and model a special case of asset management problems. Thus, based on some conditions of stability, we herein give some stability theorems of the model.

Keywords: *Optimal control, uncertain theory, uncertain differential equation, uncertain process, stability*

1 Introduction

Liu (2007) founded uncertainty theory and was refined by Liu (2016) to study uncertainty in human behavior. Liu (2009) also developed a canonical process in the concept of uncertainty theory which is a type of stationary independent increment process, a counterpart of wiener process but the increments are normal uncertain variables and not random variables. In addition, virtually all sample paths of canonical process are Lipschitz continuous.

Based on canonical process, Liu (2008) proposed uncertain differential equation where Chen and Liu (2010) gave the existence and uniqueness of solutions for an uncertain differential equation and Yao et al gave some stability theorems of uncertain differential equation. So far, uncertain differential equation has been used in modeling real life problems like stock model (Liu 2009; Peng and Yao 2011; Chen 2011) and in optimization method like optimal control (Zhu 2010; Deng and Zhu 2012; Latunde and Bamigbola 2016).

In this paper, we state some theorems of stability to the proposed model and optimal control of uncertainty in the management of capital asset based on some stability conditions of uncertain differential equation.

2 Preliminaries

Uncertainty theory is a branch of mathematics for modeling belief degrees. This theory is based on some concepts which may be referred to Liu (2016). For easy interpretation, some of the concepts are given.

Let Γ be a nonempty set and L a σ - algebra over Γ such that (Γ, L) be a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 2.1 (Liu 2007): A set function M defined on the σ-algebra over L is called an uncertain measure if it satisfies the following axioms:

Axiom 1. *(Normality Axiom)*: $M\{\Lambda\} = 1$ *for the universal set* Γ.

Axiom 2. *(Duality Axiom)*: $M\{\Lambda\} + M\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Subadditivity Axiom)*: For every countable sequence of events, $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{2.1}$$

Axiom 4. *(Product Axiom)*: Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1,2,\cdots$ The product uncertain measure M is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \min_{1\le k\le\infty} M_k\{\Lambda_k\} \tag{2.2}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1,2,\cdots,$ respectively.

Definition 2.2 (Liu 2009): Let (Γ, L, M) be an uncertainty space and let T be a totally ordered ser (e.g time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t.

Definition 2.3 (Liu 2009): An uncertain process C_σ is said to be a canonical Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_σ has stationary and independent increments,

(iii) every increment $C_{s+\sigma} - C_s$ is a normal uncertain variable with expected value 0 and variance σ^2. The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right]^{-1}, \quad x \in \Re \tag{2.3}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi}\ln\frac{y}{1-y}, \quad y \in \Re \tag{2.4}$$

Definition 2.3 (Liu 2007): Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}dx - \int_{-\infty}^{0} M\{\xi \le x\}dx \tag{2.5}$$

provided that at least one of the two integrals is finite

Definition 2.4 (Liu 2008): An uncertain process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where $t_1, t_2, \cdots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. It is noted that the increments are also independent of the initial state.

Definition 2.5 (Liu 2008): Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.6}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (2.3) and

(2.4) identically in t.

Definition 2.6 (Liu 2008): Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of X_t.

Definition 2.7 (Liu 2016): An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t.

3 Uncertain optimal control problem in asset management

Suppose an individual invests his wealth in capital asset of his business A_t from time t_0 to time t_n. He starts with a known net worth X_0. At time t, he must choose what fraction of his net worth to utilize on capital asset, ψ, what fraction of his net worth is incurred on liability of the business, τ and thus, determine the expected present net asset, E such that the net worth is maximized.

Table 3.1 Definition of Parameters

Parameter	Description
X_t	Net worth at time (state variable) t
K_t	Consumption at time t
A_t	Capital asset at time t
I_t	Investment at time t
T_t	Indirect tax at time t
D_t	Depreciation at time t
Z_t	Net Foreign supply at time t (less home supply from foreign supply)
L_t	Liability at time t
R_t	Net foreign factor revenue generated at time t
b_t	Return on capital asset at time t
τ	Liability ratio (control) $\tau > -1$
σ_r	Diffusion volatility of liability (with variance σ_r^2 per unit time)
ψ	Capital asset ratio (control) $\psi > 0$
σ_b	Diffusion volatility of asset (with variance σ_b^2 per unit time)
α	Capital gain on asset due to inflation
σ_p	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
β	Mean rate of return on asset
ω	Mean interest rate of liability
C_t	Canonical process
μ	Consumption level
f	Investment ratio
j	Tax ratio
g	Depreciation ratio

h	Net foreign asset ratio
η	discount rate
λ	degree of relative risk
U	Utility function
E	Expectation operator

Therefore, a dynamic optimization model of the expected present value of asset over a given the life cycle is herein presented following the study of portfolio selection by Merton (1971). It is assumed that the goal of the asset management is to choose the optimal utilization and asset allocation policies for maximizing a value function which discounts exponentially future uncertain values of Hyperbolic Absolute Risk Aversion (HARA) utility function over a given time horizon with net worth of tangible assets as the state variable.

The risky asset is assumed to earn an uncertain return and an uncertain gain with mean rate of return and capital gain. Furthermore, we express the change in liability as sum of liability service with an assumption of uncertainty, consumptions, investment and net foreign supply, less taxation, depreciation and revenue over a period of time. Thus, we have

$$J(X) = \max_{\psi} E_C\left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X_t)^{\lambda}\, dt\right] \tag{3.1}$$

subject to

$$dX_t = [(\alpha+\beta)\psi - (\omega(\psi-1)+\mu+f+h-j-g)]X_t dt + [\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t dC_t$$

Latunde and Bamigbola (2016) (3.2)

4 Stability analysis

We prove the stability of the model by showing that the constraint (4.19) is stable using the following conditions:

1. Suppose the coefficients $f(t,x)$ and $g(t,x)$ of $dX_t = f(t,x)dt + g(t,x)dC$ are continuous on $[0,+\infty)\times\Re$. Then the solution of an uncertain differential equation X_t is said to be stable if for any $\varepsilon > 0$ and $\rho > 0$, $\exists$ a $\Psi = \Psi(\varepsilon,\rho) > 0$ such that for any solution Y_t with $|Y_0 - X_0| < \Psi$, we have

$$M\{\sup_{t\geq 0} |Y_t - X_t| < \varepsilon\} > 1-\rho \quad \text{Gao(2010)}$$

2. An uncertain differential equation is said to be stable if for any two solutions X_t and Y_t, we have

$$\lim_{|X_0 - Y_0|\to 0} M\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = 1$$

for any given number $\varepsilon > 0$. Liu (2009)

Theorem 4.1 (Extreme Value Theorem, Liu 2013): Let X_t be a sample-space continuous independent increment process with uncertainty distribution $\Phi(x)$. Then the supremum

$$\sup_{0\leq t\leq s} X_t$$

has an uncertain distribution

$$\Psi(x) = \inf_{0\le t\le s}\Phi(x)$$

and the infimum

$$\inf_{0\le t\le s} X_t$$

has an uncertain distribution

$$\Psi(x) = \sup_{0\le t\le s}\Phi(x)$$

Theorem 4.2: Suppose the coefficients $[(\alpha+\beta)\psi-(\omega(\psi-1)+\mu+f+h-j-g)] = a_{1t}$ and $[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)] = a_{2t}$ satisfy

$$\sup_{t\ge 0}\int_0^t a_{1t}\,dt = B < +\infty, \qquad \int_0^{+\infty} |a_{2t}|\,dt = G < +\infty, \text{Gao(2010)}$$

then every solution of constraint (3.2) is bounded.

Proof:

Suppose X_t is the solution to the constraint (3.2) with initial value X_0. Therefore, from

$$X_t = X_0\exp\left(\int_{t_0}^{t_n}[(\alpha+\beta)\psi-(\omega(\psi-1)+\mu+f+h-j-g)]dt\right.$$

$$\left.+\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t\right)$$

we obtain

$$\sup_{t\ge 0}|X_t| = \sup_{t\ge 0}|X_0|\exp\left(\int_{t_0}^{t_n}[(\alpha+\beta)\psi-(\omega(\psi-1)+\mu+f+h-j-g)]dt\right.$$

$$\left.+\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t\right)$$

$$\le |X_0|\exp(\mathrm{B})\exp\left(\sup_{t\ge 0}\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t\right)$$

Since

$$\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t : N\left(0,\int_{t_0}^{t_n}|\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)|\,dt\right)$$

which implies

$$\int_{t_0}^{t_n}|\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)|\,dt \le \int_{t_0}^{+\infty}|\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)|\,dt = G < +\infty,$$

Thus, by theorem (4.1), we obtain

$$M\left\{\sup_{t\ge 0}\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t \le l\right\} = \left[1+\exp\left(\frac{-\pi l}{\sqrt{3}G}\right)\right]^{-1}, \quad \forall l \ge 0$$

Thus, $\exists$ a positive number θ for any $\rho > 0$ such that for any $t > 0$, we have

$$M\left\{\sup_{t\ge 0}\int_{t_0}^{t_n}[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]dC_t < \theta\right\} > 1-\rho$$

Putting $\Omega = |X_0|\exp(\mathrm{B})\exp(\theta)$, we have

$$M\left\{\sup_{t\ge 0}|X_t| < \Omega\right\} > 1-\rho$$

which implies that every solution of the systems (3.1) and (3.2) is bounded

Theorem 4.3: The zero solution of the system (3.1) and (3.2) is stable, if and only if every solution of constraint (3.2) is bounded.

Proof:

Since every solution of the constraint (3.2) is bounded. Let X_t be a solution with initial value $X_0 \neq 0$.

Boundedness implies for any $\rho > 0$, $\exists$ an $\Omega = \Omega(\rho, X_t > 0)$ such that $M\left\{\sup_{t\geq 0} |X_t| < \Omega\right\} > 1-\rho.$

For any $\varepsilon > 0$ and $\rho > 0$, let $\Psi = \dfrac{|X_0|\varepsilon}{\Omega}$.

We therefore assume Y_t to be another solution with initial value Y_0, where $|Y_0 - 0| < \varepsilon$. From

$$|Y_t| = |Y_0| \exp\left(\int_{t_0}^{t_n} [(\alpha+\beta)\psi - (\omega(\psi-1) + \mu + f + h - j - g)]dt\right.$$
$$\left. + \int_{t_0}^{t_n} [\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]dC_t\right) = \frac{|Y_0|}{|X_0|}|X_t|,$$

we have

$$M\{\sup_{t\geq 0} |Y_t - 0| < \varepsilon\} = M\left\{\sup_{t\geq 0} |X_t| < \frac{|X_0|}{|Y_0|}\varepsilon\right\} \geq M\left\{\sup_{t\geq 0} |X_t| < \Omega\right\} > 1-\rho$$

which implies that the zero solution is stable.

Suppose the zero solution of the systems (3.1) and (3.2) is stable. Let X_t be the solution to the uncertain equation with initial value $X_0 > 0$. According to the stability condition, for any $\varepsilon > 0$ and $\rho > 0$, $\exists$ an $\Omega = \Omega(\varepsilon, \psi) > 0$ such that whenever $|X_0| < \Psi$, we have $M\left\{\sup_{t\geq 0} |X_t| < \varepsilon\right\} > 1-\rho.$

Let $\Omega = \dfrac{Y_0\varepsilon}{X_0}$ for a solution Y_t. Therefore,

$$M\{\sup_{t\geq 0} |Y_t| < \Omega\} = M\left\{\sup_{t\geq 0} |X_t| < \frac{|X_0|}{|Y_0|}\Omega\right\} = M\left\{\sup_{t\geq 0} |X_t| < \varepsilon\right\} > 1-\rho$$

which implies that every solution of the system is bounded. Hence the completion of the proof.

Theorem 4.4 (Yao et al.,2013): Let C_t be a canonical process on an uncertain space (Γ, L, M) and $k(\gamma)$ be the least Lipschitz constant of sample path $C_t(\gamma)$. Then

$$M\{k < +\infty\} = 1$$

Theorem 4.5 (Yao et al.,2013): The uncertain differential equation $dX_t = f(t, X_t)dt + g(t, X_t)dC_t$ is stable if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the linear condition

$$|f(t,x)| + |g(t,x)| \leq K(1+|x|), \quad \forall x \in \Re, t \geq 0$$

for some constant K and strong Lipschitz condition

$$|f(t,x)| - |f(t,y)| + |g(t,x)| - |g(t,y)| \leq L(t)|x-y|, \quad \forall x, y \in \Re, t \geq 0$$

for some bounded and integrable function $L(t)$ on $[0, +\infty)$

Theorem 4.6: The systems (3.1) and (3.2) is stable if the coefficients $f(t,x)$ and $g(t,x)$ satisfy the linear condition

$$|f(t,x)|+|g(t,x)|\leq K(1+|x|), \quad \forall x\in\Re, t\geq 0$$

for some constant K and strong Lipschitz condition

$$|f(t,x)|-|f(t,y)|+|g(t,x)|-|g(t,y)|\leq L(t)|x-y|, \quad \forall x,y\in\Re, t\geq 0$$

for some bounded and integrable function $L(t)$ on time interval $t_0<t<t_n$.

Proof:

Let X_0 and Y_0 represent two solutions with X_0 and Y_0 as initial values respectively.

Thus for each γ, we have

$$d|X_t(\gamma)-Y_t(\gamma)|\leq|[(\alpha+\beta)\psi-(\omega(\psi-1)+\mu+f+h-j-g)]X_t(\gamma)$$

$$-[(\alpha+\beta)\psi-(\omega(\psi-1)+\mu+f+h-j-g)]Y_t(\gamma)|$$

$$+|[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]X_t(\gamma)-[\psi\sigma_p+\psi\sigma_b-\psi(\sigma_r-1)]Y_t(\gamma)|$$

$$\leq L(t)|X_t(\iota)-Y_t(\gamma)|dt+L(t)k(\gamma)|X_t(\gamma)-Y_t(\gamma)|dt$$

$$=L(t)(1+k(\gamma))|X_t(\gamma)-Y_t(\gamma)|dt$$

where k represents the Lipschitz constant on the sample path $C_t(\gamma)$. Thereafter, by the Grownwall's equality, we have

$$|X_t(\gamma)-Y_t(\gamma)|\leq|X_0-Y_0|\exp\left((1+k(\gamma))\int_{t_0}^{t_n}L(t)dt\right)$$

Therefore, for any $\varepsilon>0$, we have

$$M\{|X_t-Y_t|<\varepsilon \quad \forall t\geq 0\}$$

$$\leq M\left\{|X_0-Y_0|\exp\left((1+k(\gamma))\int_{t_0}^{t_n}L(t)dt\right)<\varepsilon\right\}$$

By theorem (4.4), we get

$$M\left\{|X_0-Y_0|\exp\left((1+k(\gamma))\int_{t_0}^{t_n}L(t)dt\right)<\varepsilon\right\}\to 1$$

Thus, as $|X_0-Y_0|\to 0$ we obtain

$$\lim_{|X_0-Y_0|\to 0}M\{|X_t-Y_t|<\varepsilon \quad \forall t\geq 0\}=1$$

Hence, the proof.

5 Conclusion

Uncertain differential equation was used to model a type of asset management problem where an investor is interested in determining the expected present net asset such that the net worth is maximized.

This paper gave some stability theorems and proofs of uncertain optimal control model of management of net risky capital asset based on the necessary and sufficient conditions for uncertain differential equations being stable.

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